

Further mathematics
Higher level
Paper 2

Thursday 11 May 2017 (morning)

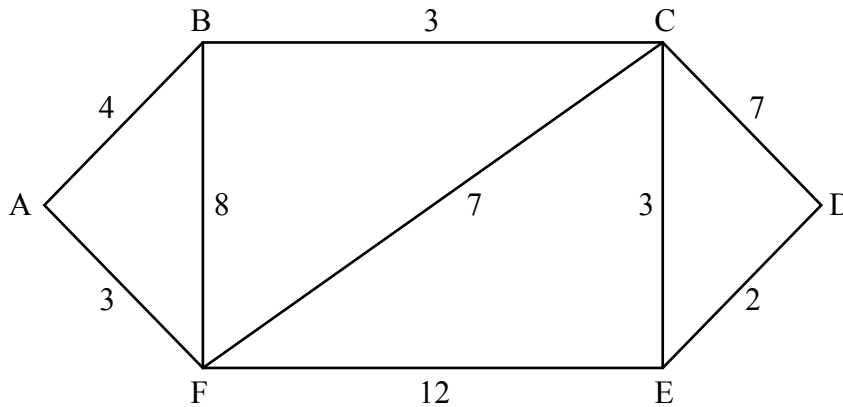
2 hours 30 minutes

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[150 marks]**.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 11]



The diagram shows the graph G with the weights of the edges marked.

- (a) (i) State what features of the graph enable you to state that G contains an Eulerian trail but no Eulerian circuit. [4]
- (ii) Write down an Eulerian trail. [4]
- (b) Use Dijkstra's algorithm to find the path of minimum total weight joining A to D , stating this total weight. Your solution should show clearly that this algorithm has been used. [7]

2. [Maximum mark: 15]

- (a) The group $\{G, *\}$ has a subgroup $\{H, *\}$. The relation R is defined such that for $x, y \in G$, xRy if and only if $x^{-1}*y \in H$. Show that R is an equivalence relation. [8]
- (b) Consider the special case in which $G = \{1, 3, 4, 9, 10, 12\}$, $H = \{1, 12\}$ and $*$ denotes multiplication modulo 13.
 - (i) Show that $3R10$.
 - (ii) Determine the three equivalence classes. [7]

3. [Maximum mark: 15]

(a) The sequence $\{u_n\}$ satisfies the second-degree recurrence relation

$$u_{n+2} = u_{n+1} + 6u_n, n \in \mathbb{Z}^+.$$

(i) Write down the auxiliary equation.

(ii) Given that $u_1 = 12, u_2 = 6$, show that

$$u_n = 2 \times 3^n - 3 \times (-2)^n.$$

(iii) Determine the value of $\lim_{n \rightarrow \infty} \frac{u_n + u_{n-1}}{u_n - u_{n-1}}$. [11]

(b) Another sequence $\{v_n\}$ is such that

$$v_n = u_{2n}, n \in \mathbb{Z}^+.$$

Determine the second-degree recurrence relation satisfied by $\{v_n\}$. [4]

Turn over

4. [Maximum mark: 19]

(a) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} \lambda & 3 & 2 \\ 2 & 4 & \lambda \\ 3 & 7 & 3 \end{bmatrix}.$$

- (i) Find an expression for $\det(\mathbf{A})$ in terms of λ , simplifying your answer.
- (ii) Hence show that \mathbf{A} is singular when $\lambda = 1$ and find the other value of λ for which \mathbf{A} is singular. [5]

Suppose now that $\lambda = 1$ so consider the matrix

$$\mathbf{B} = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 1 \\ 3 & 7 & 3 \end{bmatrix}.$$

- (b) (i) Explain how it can be seen immediately that \mathbf{B} is singular without calculating its determinant.
- (ii) Determine the null space of \mathbf{B} .
- (iii) Explain briefly how your results verify the rank-nullity theorem. [7]
- (c) Prove, using mathematical induction, that

$$\mathbf{B}^n = 8^{n-2} \mathbf{B}^2 \text{ for } n \in \mathbb{Z}^+, n \geq 3. \quad [7]$$

5. [Maximum mark: 24]

(a) (i) By considering integration as the reverse of differentiation, show that for

$$0 \leq x < \frac{\pi}{2}$$

$$\int \sec x dx = \ln(\sec x + \tan x) + C.$$

(ii) Hence, using integration by parts, show that

$$\int \sec^3 x dx = \frac{1}{2}(\sec x \tan x + \ln(\sec x + \tan x)) + C. \quad [8]$$

(b) Consider the differential equation

$$\frac{dy}{dx} + y \tan x = 2 \sec^2 x, \quad 0 \leq x < \frac{\pi}{2}, \quad \text{given that } y = 1 \text{ when } x = 0.$$

(i) Find an integrating factor and hence solve the differential equation, giving your answer in the form $y = f(x)$.

(ii) Starting with the differential equation, show that

$$\frac{d^2 y}{dx^2} + y = 2 \sec^2 x \tan x.$$

(iii) Hence, by using your calculator to draw two appropriate graphs or otherwise, find the x -coordinate of the point of inflection on the graph of $y = f(x)$. [16]

Turn over

6. [Maximum mark: 14]

(a)

Figure 1

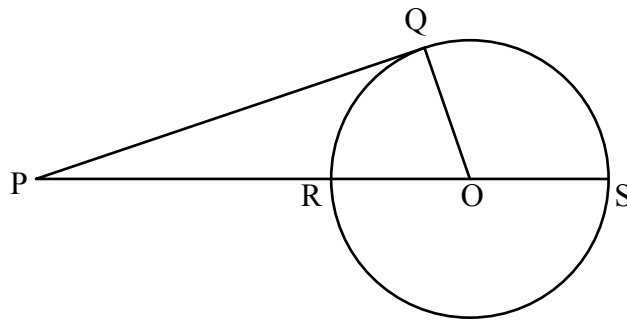


Figure 1 shows a tangent [PQ] at the point Q of a circle and a line [PS] meeting the circle at the points R, S and passing through the centre O of the circle.

- (i) Show that $PQ^2 = PR \times PS$.
- (ii) State briefly how this result can be generalized to give the tangent-secant theorem.

[4]

(b)

Figure 2

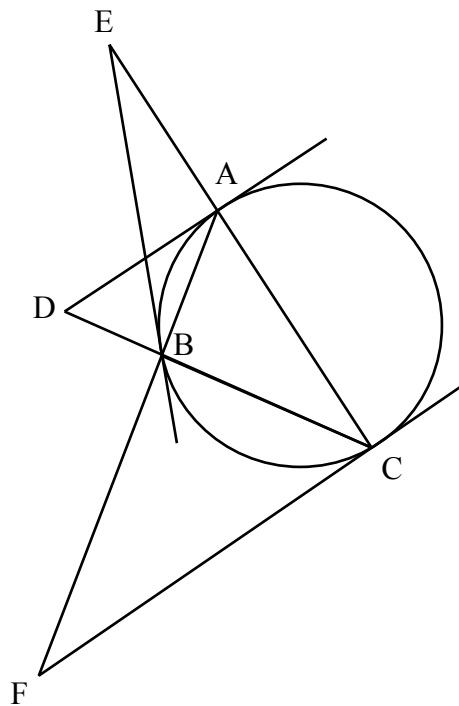


Figure 2 shows a triangle ABC inscribed in a circle. The tangents at the points A, B, C meet the opposite sides of the triangle externally at the points D, E, F respectively.

(This question continues on the following page)

(Question 6 continued)

(i) Show that $\frac{AD^2}{BD^2} = \frac{CD}{BD}$.

(ii) By considering a pair of similar triangles, show that

$$\frac{AD}{BD} = \frac{AC}{AB} \text{ and hence that } \frac{CD}{BD} = \frac{AC^2}{AB^2}.$$

(iii) By writing down and using two further similar expressions, show that the points D, E, F are collinear.

[10]

7. [Maximum mark: 18]

The discrete random variable X has the following probability distribution.

$$P(X = x) = \frac{kx}{3^x} \text{ where } x \in \mathbb{Z}^+ \text{ and } k \text{ is a constant.}$$

(a) (i) Write down the first three terms of the infinite series for $G(t)$, the probability generating function for X .

(ii) Determine the radius of convergence of this infinite series.

(iii) By considering $\left(1 - \frac{t}{3}\right)G(t)$, show that

$$G(t) = \frac{3kt}{(3-t)^2}.$$

(iv) Hence show that $k = \frac{4}{3}$.

[10]

(b) (i) Show that $\ln G(t) = \ln 4 + \ln t - 2 \ln(3 - t)$.

(ii) By differentiating both sides of this equation, determine the values of $G'(1)$ and $G''(1)$.

(iii) Hence find $\text{Var}(X)$.

[8]

Turn over

8. [Maximum mark: 17]

The set $S_n = \{1, 2, 3, \dots, n-2, n-1\}$, where n is a prime number greater than 2, and \times_n denotes multiplication modulo n .

- (a) (i) Show that there are no elements $a, b \in S_n$ such that $a \times_n b = 0$.
- (ii) Show that, for $a, b, c \in S_n$, $a \times_n b = a \times_n c \Rightarrow b = c$. [4]
- (b) Show that $G_n = \{S_n, \times_n\}$ is a group. You may assume that \times_n is associative. [4]
- (c) (i) Show that the order of the element $(n-1)$ is 2.
- (ii) Show that the inverse of the element 2 is $\frac{1}{2}(n+1)$.
- (iii) Explain why the inverse of the element 3 is $\frac{1}{3}(n+1)$ for some values of n but not for other values of n .
- (iv) Determine the inverse of the element 3 in G_{11} .
- (v) Determine the inverse of the element 3 in G_{31} . [9]

9. [Maximum mark: 17]

- (a) The point (x, y) is rotated through an anticlockwise angle α about the origin to become the point (X, Y) . Assume that the rotation can be represented by

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Show, by considering the images of the points $(1, 0)$ and $(0, 1)$ under this rotation that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}. \quad [3]$$

- (b) The hyperbola with equation $x^2 - 4xy - 2y^2 = 3$ is rotated through an acute anticlockwise angle α about the origin.
- (i) By expressing (x, y) in terms of (X, Y) , determine the equation of the rotated hyperbola in terms of X and Y .
- (ii) Verify that the coefficient of XY in the equation is zero when $\tan \alpha = \frac{1}{2}$.
- (iii) Determine the equation of the rotated hyperbola in this case, giving your answer in the form $\frac{X^2}{A^2} - \frac{Y^2}{B^2} = 1$.
- (iv) Hence find the coordinates of the foci of the hyperbola prior to rotation. [14]